In this appendix we give several useful mathematical facts. We begin with some combinatorial definitions and facts.

Logarithms and Exponents

The logarithm function is defined as

$$\log_b a = c \quad \text{if} \quad a = b^c.$$ 

The following identities hold for logarithms and exponents:

1. $$\log_b(ac) = \log_b a + \log_b c$$
2. $$\log_b(a/c) = \log_b a - \log_b c$$
3. $$\log_b a^c = c \log_b a$$
4. $$\log_b a = (\log_c a) / (\log_c b)$$
5. $$b^{\log_b a} = a^{\log_c b}$$
6. $$(b^a)^c = b^{ac}$$
7. $$b^a b^c = b^{a+c}$$
8. $$b^a / b^c = b^{a-c}$$

In addition, we have the following:

**Proposition A.1:** If $$a > 0$$, $$b > 0$$, and $$c > a + b$$, then

$$\log a + \log b < 2\log c - 2.$$ 

**Justification:** It is enough to show that $$ab < c^2/4$$. We can write

$$ab = \frac{a^2 + 2ab + b^2 - a^2 + 2ab - b^2}{4} = \frac{(a + b)^2 - (a - b)^2}{4} \leq \frac{(a + b)^2}{4} < \frac{c^2}{4}.$$ 

The natural logarithm function $$\ln x = \log_e x$$, where $$e = 2.71828\ldots$$, is the value of the following progression:

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots.$$
In addition,

\[ e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \]

\[ \ln(1 + x) = x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \cdots. \]

There are a number of useful inequalities relating to these functions (which derive from these definitions).

**Proposition A.2:** If \( x > -1 \),

\[ \frac{x}{1+x} \leq \ln(1+x) \leq x. \]

**Proposition A.3:** For \( 0 \leq x < 1 \),

\[ 1+x \leq e^x \leq \frac{1}{1-x}. \]

**Proposition A.4:** For any two positive real numbers \( x \) and \( n \),

\[ \left(1 + \frac{x}{n}\right)^n \leq e^x \leq \left(1 + \frac{x}{n}\right)^{n+x/2}. \]

**Integer Functions and Relations**

The “floor” and “ceiling” functions are defined respectively as follows:

1. \([x]\) = the largest integer less than or equal to \( x \).
2. \(\lceil x \rceil\) = the smallest integer greater than or equal to \( x \).

The modulo operator is defined for integers \( a \geq 0 \) and \( b > 0 \) as

\[ a \mod b = a - \left\lfloor \frac{a}{b} \right\rfloor b. \]

The factorial function is defined as

\[ n! = 1 \cdot 2 \cdot 3 \cdots (n-1)n. \]

The binomial coefficient is

\[ \binom{n}{k} = \frac{n!}{k!(n-k)!}, \]

which is equal to the number of different combinations one can define by choosing \( k \) different items from a collection of \( n \) items (where the order does not matter).

The name “binomial coefficient” derives from the binomial expansion:

\[ (a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}. \]

We also have the following relationships.
Appendix A. Useful Mathematical Facts

**Proposition A.5:** If $0 \leq k \leq n$, then

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \frac{n^k}{k!}.$$  

**Proposition A.6 (Stirling’s Approximation):**

$$n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n} + \varepsilon(n)\right),$$

where $\varepsilon(n)$ is $O(1/n^2)$.

The **Fibonacci progression** is a numeric progression such that $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

**Proposition A.7:** If $F_n$ is defined by the Fibonacci progression, then $F_n$ is $\Theta(g^n)$, where $g = (1 + \sqrt{5})/2$ is the so-called **golden ratio**.

**Summations**

There are a number of useful facts about summations.

**Proposition A.8:** Factoring summations:

$$\sum_{i=1}^{n} af(i) = a \sum_{i=1}^{n} f(i),$$

provided $a$ does not depend upon $i$.

**Proposition A.9:** Reversing the order:

$$\sum_{i=1}^{n} \sum_{j=1}^{m} f(i,j) = \sum_{j=1}^{m} \sum_{i=1}^{n} f(i,j).$$

One special form of is a **telescoping sum**:

$$\sum_{i=1}^{n} (f(i) - f(i-1)) = f(n) - f(0),$$

which arises often in the amortized analysis of a data structure or algorithm.

The following are some other facts about summations that arise often in the analysis of data structures and algorithms.

**Proposition A.10:** $\sum_{i=1}^{n} i = n(n+1)/2$.

**Proposition A.11:** $\sum_{i=1}^{n} i^2 = n(n+1)(2n+1)/6$. 
Proposition A.12: If $k \geq 1$ is an integer constant, then
\[ \sum_{i=1}^{n} i^k \text{ is } \Theta(n^{k+1}). \]

Another common summation is the geometric sum, $\sum_{i=0}^{n} a^i$, for any fixed real number $0 < a \neq 1$.

Proposition A.13:
\[ \sum_{i=0}^{n} a^i = \frac{a^{n+1} - 1}{a - 1}, \]
for any real number $0 < a \neq 1$.

Proposition A.14:
\[ \sum_{i=0}^{\infty} a^i = \frac{1}{1 - a} \]
for any real number $0 < a < 1$.

There is also a combination of the two common forms, called the linear exponential summation, which has the following expansion:

Proposition A.15: For $0 < a \neq 1$, and $n \geq 2$,\[ \sum_{i=1}^{n} ia^i = \frac{a - (n+1)a^{n+1} + na^{n+2}}{(1-a)^2}. \]

The $n^{th}$ Harmonic number $H_n$ is defined as
\[ H_n = \sum_{i=1}^{n} \frac{1}{i}. \]

Proposition A.16: If $H_n$ is the $n^{th}$ harmonic number, then $H_n$ is $\ln n + \Theta(1)$.

Basic Probability

We review some basic facts from probability theory. The most basic is that any statement about a probability is defined upon a sample space $S$, which is defined as the set of all possible outcomes from some experiment. We leave the terms “outcomes” and “experiment” undefined in any formal sense.

Example A.17: Consider an experiment that consists of the outcome from flipping a coin five times. This sample space has $2^5$ different outcomes, one for each different ordering of possible flips that can occur.

Sample spaces can also be infinite, as the following example illustrates.
Example A.18: Consider an experiment that consists of flipping a coin until it comes up heads. This sample space is infinite, with each outcome being a sequence of \(i\) tails followed by a single flip that comes up heads, for \(i = 1, 2, 3, \ldots\).

A probability space is a sample space \(S\) together with a probability function \(Pr\) that maps subsets of \(S\) to real numbers in the interval \([0, 1]\). It captures mathematically the notion of the probability of certain “events” occurring. Formally, each subset \(A\) of \(S\) is called an event, and the probability function \(Pr\) is assumed to possess the following basic properties with respect to events defined from \(S\):

1. \(Pr(\emptyset) = 0\).
2. \(Pr(S) = 1\).
3. \(0 \leq Pr(A) \leq 1\), for any \(A \subseteq S\).
4. If \(A, B \subseteq S\) and \(A \cap B = \emptyset\), then \(Pr(A \cup B) = Pr(A) + Pr(B)\).

Two events \(A\) and \(B\) are independent if

\[
Pr(A \cap B) = Pr(A) \cdot Pr(B).
\]

A collection of events \(\{A_1, A_2, \ldots, A_n\}\) is mutually independent if

\[
Pr(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = Pr(A_{i_1}) \cdot Pr(A_{i_2}) \cdot \cdots \cdot Pr(A_{i_k}),
\]

for any subset \(\{A_{i_1}, A_{i_2}, \ldots, A_{i_k}\}\).

The conditional probability that an event \(A\) occurs, given an event \(B\), is denoted as \(Pr(A|B)\), and is defined as the ratio

\[
\frac{Pr(A \cap B)}{Pr(B)},
\]

assuming that \(Pr(B) > 0\).

An elegant way for dealing with events is in terms of random variables. Intuitively, random variables are variables whose values depend upon the outcome of some experiment. Formally, a random variable is a function \(X\) that maps outcomes from some sample space \(S\) to real numbers. An indicator random variable is a random variable that maps outcomes to the set \(\{0, 1\}\). Often in data structure and algorithm analysis we use a random variable \(X\) to characterize the running time of a randomized algorithm. In this case, the sample space \(S\) is defined by all possible outcomes of the random sources used in the algorithm.

We are most interested in the typical, average, or “expected” value of such a random variable. The expected value of a random variable \(X\) is defined as

\[
E(X) = \sum_x x Pr(X = x),
\]

where the summation is defined over the range of \(X\) (which in this case is assumed to be discrete).
Appendix: Useful Mathematical Facts

Proposition A.19 (The Linearity of Expectation): Let $X$ and $Y$ be two random variables and let $c$ be a number. Then

$$E(X + Y) = E(X) + E(Y) \quad \text{and} \quad E(cX) = cE(X).$$

Example A.20: Let $X$ be a random variable that assigns the outcome of the roll of two fair dice to the sum of the number of dots showing. Then $E(X) = 7$.

Justification: To justify this claim, let $X_1$ and $X_2$ be random variables corresponding to the number of dots on each die. Thus, $X_1 = X_2$ (i.e., they are two instances of the same function) and $E(X) = E(X_1 + X_2) = E(X_1) + E(X_2)$. Each outcome of the roll of a fair die occurs with probability $1/6$. Thus,

$$E(X_i) = \frac{1}{6} + \frac{2}{6} + \frac{3}{6} + \frac{4}{6} + \frac{5}{6} + \frac{6}{6} = \frac{7}{2},$$

for $i = 1, 2$. Therefore, $E(X) = 7$.

Two random variables $X$ and $Y$ are independent if

$$\Pr(X = x | Y = y) = \Pr(X = x),$$

for all real numbers $x$ and $y$.

Proposition A.21: If two random variables $X$ and $Y$ are independent, then

$$E(XY) = E(X)E(Y).$$

Example A.22: Let $X$ be a random variable that assigns the outcome of a roll of two fair dice to the product of the number of dots showing. Then $E(X) = 49/4$.

Justification: Let $X_1$ and $X_2$ be random variables denoting the number of dots on each die. The variables $X_1$ and $X_2$ are clearly independent; hence

$$E(X) = E(X_1X_2) = E(X_1)E(X_2) = (7/2)^2 = 49/4.$$  

The following bound and corollaries that follow from it are known as Chernoff bounds.

Proposition A.23: Let $X$ be the sum of a finite number of independent $0/1$ random variables and let $\mu > 0$ be the expected value of $X$. Then, for $\delta > 0$,

$$\Pr(X > (1+\delta)\mu) < \left[ \frac{e^\delta}{(1+\delta)^{1+\delta}} \right]^\mu.$$
Useful Mathematical Techniques

To compare the growth rates of different functions, it is sometimes helpful to apply the following rule.

Proposition A.24 (L’Hôpital’s Rule): If we have \( \lim_{n \to \infty} f(n) = +\infty \) and we have \( \lim_{n \to \infty} g(n) = +\infty \), then \( \lim_{n \to \infty} \frac{f(n)}{g(n)} = \lim_{n \to \infty} \frac{f'(n)}{g'(n)} \), where \( f'(n) \) and \( g'(n) \) respectively denote the derivatives of \( f(n) \) and \( g(n) \).

In deriving an upper or lower bound for a summation, it is often useful to split a summation as follows:

\[
\sum_{i=1}^{n} f(i) = \sum_{i=1}^{j} f(i) + \sum_{i=j+1}^{n} f(i).
\]

Another useful technique is to bound a sum by an integral. If \( f \) is a nondecreasing function, then, assuming the following terms are defined,\[
\int_{a-1}^{b} f(x) \, dx \leq \sum_{i=a}^{b} f(i) \leq \int_{a}^{b+1} f(x) \, dx.
\]

There is a general form of recurrence relation that arises in the analysis of divide-and-conquer algorithms:

\[
T(n) = aT(n/b) + f(n),
\]

for constants \( a \geq 1 \) and \( b > 1 \).

Proposition A.25: Let \( T(n) \) be defined as above. Then

1. If \( f(n) = O(n^{\log_b a - \epsilon}) \), for some constant \( \epsilon > 0 \), then \( T(n) = \Theta(n^{\log_b a}) \).
2. If \( f(n) = \Theta(n^{\log_b a \log^k n}) \), for a fixed nonnegative integer \( k \geq 0 \), then \( T(n) = \Theta(n^{\log_b a \log^{k+1} n}) \).
3. If \( f(n) = \Omega(n^{\log_b a + \epsilon}) \), for some constant \( \epsilon > 0 \), and if \( af(n/b) \leq cf(n) \), then \( T(n) = \Theta(f(n)) \).

This proposition is known as the master method for characterizing divide-and-conquer recurrence relations asymptotically.